Unifying treatment of nonequilibrium and unstable dynamics of cold bosonic atom system with time-dependent order parameter in Thermo Filed Dynamics

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The coupled equations which describe the temporal evolution of the Bose-Einstein condensed system are derived in the framework of nonequilibrium Thermo Field Dynamics. The key element is that they are not the naive assemblages of presumable equations, but are the self-consistent ones derived by appropriate renormalization conditions. While the order parameter is time-dependent, an explicit quasiparticle picture is constructed by a time-dependent expansion. Our formulation is valid even for the system with a unstable condensate, and describes the condensate decay caused by the Landau instability as well as by the dynamical one.

I. INTRODUCTION

The systems of trapped cold atoms are ideal for studying the foundations of quantum many-body theories such as quantum field theory and thermal field theory. They are dilute and weak-interacting, so theoretical calculations can be compared with experimental results directly. Since the realization of Bose–Einstein condensates [1–3], many intriguing phenomena have been observed with good accuracy, and offer opportunities to test many aspects of quantum many-body theories in both equilibrium and nonequilibrium. Among them, the unstable phenomena of the condensate attract our attention, because firstly to formulate unstable quantum many-body systems is still an open problem and and secondly nonequilibrium processes accompany the instability in thermal situation.

Theoretically, the instability of the condensate is characterized by the eigenvalue of the Bogoliubov-de Gennes (BdG) equations [4–6], which follow from linearization of the time-dependent Gross-Pitaevskii (TDGP) equation [7]. Since the BdG equations are generally eigenvalue ones for non-Hermitian matrices , their eigenvalues can be complex. The emergence of complex eigenvalues is interpreted as the sign of the dynamical instability. This instability is associated with the decay of the initial configuration of the condensate and can occur even at zero temperature. On the other hand, if the negative eigenvalues for a positive-norm mode are present, the system shows another instability, called the Landau instability. This instability, in which the thermal cloud plays an essential role to drive the condensate toward a lower energy state, is suppressed at very low temperature.

The observations of both the Landau and dynamical instabilities are reported in several systems, especially in the system where the condensate flows in an optical lattice [8, 9], and they are in good agreement with the analyses of the BdG equations [10, 11].

Although the TDGP equation outlines the experiments corresponding to the dynamical instability at very low temperature [13, 14], e.g. the multiply-quantized vortex splitting [12], an detailed description of the unstable dynamics in thermal situations is not trivial. That is because there is no more quasi-stable state, and so a fully nonequilibrium theory is required.

There are known two nonequilibrium thermal field theories, i.e., the closed time path (CTP) formalism [15] and the Thermo Field Dynamics (TFD) [16]. The CTP formalism is widely used. But we employ the TFD formalism in this paper, because the concept of quasiparticle picture which is essential for quantum field theory is clear even in nonequilibrium situations. In TFD, which is a real-time canonical formalism of quantum field theory, thermal fluctuation is introduced through doubling the degrees of freedom, and the mixed state expectation in the density matrix formalism is replaced by an average of a pure state vacuum, called the thermal vacuum. It is crucial in our formulation of TFD to construct the interaction picture. In quantum field theory, the choice of unperturbed Hamiltonian and fields is that of quasiparticle picture, and concrete calculations are possible only when a particular unperturbed representation, or a particular particle picture, is specified. One does not know an exact unperturbed representation beforehand.

So far we have investigated the cold atom system with a time-independent configuration of the condensate in TFD, and derived the quantum transport equation which describes the temporal evolution of the quasiparticle number distribution [19]. It was essential to construct an explicit quasiparticle picture there. In contrast to the previous investigations [20–24] which are based on a phase-space distribution function, our transport equation contains an additional collision term which is traced back to our choice of an appropriate quasiparticle picture. The additional collision term, which we call the triple production term, vanishes in the equilibrium limit if there is no Landau instability, but remains non-vanishing to prevent the system from equilibrating if there is Landau instability. Thus our transport equation with the triple

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production term and the other ones without it predict definitely different behaviors of the unstable system.

In this paper, we derive the coupled equations which describe the nonequilibrium dynamics of the cold atom system with a time-dependent order parameter. They are the TDGP equation, the TDBdG equations, and the quantum transport equation. The key points are that while the order parameter is time-dependent, we construct a time-independent quasiparticle picture and so that the stable vacuum which are essential for quantum field theory. These are accomplished by expanding the field operator with the time-dependent complete set evaluating by the TDBdG equations [25]. The quantum correction to the TDGP equation is determined self-consistently and simultaneously as the quantum transport equation by some renormalization conditions [17]. Solving the coupled equations numerically, we illustrate the dynamics of the condensate decays with either the Landau instability or the dynamical one and discriminate the two instabilities.

This paper is organized as follows. We consider the cold bosonic atom system with a time-dependent order parameter at zero temperature in Section II. We show that it is crucial to expand the field operator by the solutions of TDBdG equations to maintain the time-dependent quasiparticle picture. In Section III, a nonequilibrium system is considered, and the degrees of freedom are doubled to treat the system with the nonequilibrium TFD. We construct a systematical method to obtain the coupled equations, and derive those explicitly in the leading order. In Section IV, we consider a simple system with the Bose–Hubbard model and calculated the coupled equations numerically. Section V is devoted to summary.

II. FORMULATION OF QUANTUM FIELD THEORY

In this section, we consider the cold bosonic atom system with a time-dependent order parameter at zero temperature. We start with the following Hamiltonian to describe the trapped dilute bosonic atoms:

$$
H = \int d^3x \Big[\psi^\dagger(x) \left(-\frac{\nabla^2}{2m} + V(\boldsymbol{x}) - \mu \right) \psi(x)
+ \frac{g}{2} \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \Big], \quad (1)
$$

where m, $V(\boldsymbol{x})$, μ, and g represent the mass of an atom, the trap potential, the chemical potential, and the coupling constant, respectively. Throughout this paper $\hbar$ is set to be unity. The bosonic field operator $\psi(x)$ obeys the canonical commutation relations

$$
\left[\psi(x), \psi^\dagger(x') \right] |_{t=t'} = \delta(\boldsymbol{x} - \boldsymbol{x}') , \quad (2)
$$

$$
\left[\psi(x), \psi(x') \right] |_{t=t'} = 0 . \quad (3)
$$

where $x = (\boldsymbol{x}, t)$. Reflecting the existence of the condensate, the field operator ψ is divided into a classical part $\zeta(x)$ and a quantum one $\varphi(x)$ on the criterion $\langle 0|\varphi(x)|0\rangle = 0$. Note that the vacuum is not yet specified and that $\zeta(x)$ is an arbitrary function at this stage, and the division must be completed later self-consistently. The doublet notation is introduced as

$$
\varphi^\alpha = \begin{pmatrix} \varphi \\ \varphi^\dagger \end{pmatrix}^\alpha , \qquad \bar\varphi^\alpha = \left(\varphi^\dagger \ \ -\varphi \right)^\alpha , \quad (4)
$$

and the unperturbed Hamiltonian H_0, bilinear and linear in φ and $\varphi^\dagger$, is

$$
H_0 = \int d^3x \Big[\frac{1}{2} \bar\varphi^\alpha T^{\alpha\beta} \varphi^\beta + \varphi^\dagger \left(h_0\zeta + \delta C \right)
+ \varphi \left(h_0\zeta^* + \delta C^* \right) \Big] , \quad (5)
$$

with

$$
T(x) = T_0(x) + \delta T(x) , \quad (6)
$$

where

$$
T_0^{\alpha\beta}(x) = \begin{pmatrix} \mathcal{L}_0(x) & \mathcal{M}_0(x) \\ -\mathcal{M}_0^*(x) & -\mathcal{L}_0(x) \end{pmatrix}^{\alpha\beta} , \quad (7)
$$

$$
\mathcal{L}_0(x) = -\frac{\nabla^2}{2m} + V(\boldsymbol{x}) - \mu + 2g|\zeta(x)|^2 , \quad (8)
$$

$$
\mathcal{M}_0(x) = g\zeta^2(x) , \quad (9)
$$

$$
h_0(x) = -\frac{\nabla^2}{2m} + V(\boldsymbol{x}) - \mu + g|\zeta(x)|^2 . \quad (10)
$$

The counter terms $\delta T(x)$ and $\delta C(x)$ are determined later self-consistently. The perturbed Hamiltonian $H_{\rm int} = H - H_0$ is given as

$$
H_{\rm int} = \int d^3x \Big[\frac{g}{2} \varphi^{\dagger,2}\varphi^2 + g\zeta^*\varphi^\dagger\varphi^2
+ g\zeta\varphi^{\dagger,2}\varphi - \varphi^\dagger\delta C - \varphi\delta C^* \Big] . \quad (11)
$$

From the original Heisenberg equation for ψ and the time-dependent ζ, the field equation for φ in the interaction picture should be

$$
i\dot\varphi = \left(\mathcal{L}_0 + \delta T^{11} \right) \varphi + \left(\mathcal{M}_0 + \delta T^{12} \right) \varphi^\dagger + h_0\zeta + \delta C - i\dot\zeta . \quad (12)
$$

Due to the last term, this time-evolution is generated not by H_0 in Eq. (5) but by

$$
H_0^\varphi = H_0 - i \int d^3x \left[\dot\zeta\varphi^\dagger - \dot\zeta^*\varphi \right] , \quad (13)
$$

as

$$
i\dot\varphi = [\varphi, H_0^\varphi] . \quad (14)
$$

The condition $\langle 0|\varphi(x)|0\rangle = 0$ must hold for any t in the unpertubed representation, which implies

$$
i\frac{\partial}{\partial t}\langle 0|\varphi(x)|0\rangle = \langle 0|i\dot\varphi|0\rangle = 0 . \quad (15)
$$

for the time-independent vacuum. According to Eq. (12), we have

$$\delta C = (i\partial_t - h_0)\zeta \,, \qquad (16)$$

and the time-evolution operator H_0^φ becomes a simple quadratic form:

$$H_0^\varphi = \frac{1}{2}\int d^3x \; \bar{\varphi}^\alpha T^{\alpha\beta}\varphi^\beta \,. \qquad (17)$$

A. Field expansion for time-independent order parameter

Before going into further discussions, let us briefly review the Bogoliubov-de Gennes (BdG) method which diagonalizes the unperturbed Hamiltonian H_0 $(= H_0^\varphi)$ in case of the time-independent order parameter $\zeta(\boldsymbol{x})$. The BdG equations are simultaneous eigenvalue equations given by [4–6]

$$T(\boldsymbol{x})\boldsymbol{y}_\ell(\boldsymbol{x}) = \omega_\ell \boldsymbol{y}_\ell(\boldsymbol{x}) \,. \qquad (18)$$

Since the operator T is non-Hermitian, the eigenvalues is not always real but can be complex in general. The condition for the emergence of complex eigenvalues in the BdG equations has been studied both numerically [32–34] and analytically [35–37], and the quantum field theoretical formulation has also been discussed [38]. The emergence of complex eigenvalues implies the dynamical instability of the system, and a drastic temporal change of the order parameter occurs then. While our propose in this subsection is to find a stable initial condition as becomes apparent later, we consider only the case where no complex eigenvalue emerges.

Eigenfunctions belonging to the non-zero real eigenvalues can be orthonormalized under the indefinite metric as

$$\int d^3x \; \boldsymbol{y}_\ell^\dagger(\boldsymbol{x})\sigma_3\boldsymbol{y}_{\ell'}(\boldsymbol{x}) = \delta_{\ell\ell'} \,, \qquad (19)$$

$$\int d^3x \; \boldsymbol{z}_\ell^\dagger(\boldsymbol{x})\sigma_3\boldsymbol{z}_{\ell'}(\boldsymbol{x}) = -\delta_{\ell\ell'} \,, \qquad (20)$$

$$\int d^3x \; \boldsymbol{y}_\ell^\dagger(\boldsymbol{x})\sigma_3\boldsymbol{z}_{\ell'}(\boldsymbol{x}) = 0 \,, \qquad (21)$$

with i-th Pauli matrix σ_i. The function $\boldsymbol{z}_\ell$, defined by $\boldsymbol{z}_\ell = \sigma_1\boldsymbol{y}_\ell^*$, is an eigenfunction belonging to $-\omega_\ell$, when $\boldsymbol{y}_\ell$ is an eigenfunction belonging to ω_ℓ.

Due to the Nanbu-Goldstone theorem [39], there is a zero mode eigenfunction in the BdG equations [30, 31]: $T\boldsymbol{y}_0 = 0$. The zero mode eigenfunction $\boldsymbol{y}_0$ is orthogonal to all the other eigenfunctions, and what is more, is orthogonal to itself. Hence, an additional adjoint mode $\boldsymbol{y}_{-1}$ has to be introduced for the completeness as

$$T\boldsymbol{y}_{-1} = I\boldsymbol{y}_0 \,, \qquad (22)$$

where I is determined to satisfy the normalization condition:

$$\int d^3x \; \boldsymbol{y}_{-1}^\dagger(\boldsymbol{x})\sigma_3\boldsymbol{y}_0(\boldsymbol{x}) = 1 \,. \qquad (23)$$

It is convenient to rewrite the whole orthonormal conditions with the 2×2 matrix form as

$$\int d^3x \; W_\Lambda(\boldsymbol{x}) \, W_{\Lambda'}^{-1}(\boldsymbol{x}) = \delta_{\Lambda\Lambda'} \,, \qquad (24)$$

where

$$W_\ell(\boldsymbol{x}) = \sigma_3 \begin{pmatrix} \boldsymbol{y}_\ell^\dagger(\boldsymbol{x}) \\ \boldsymbol{z}_\ell^\dagger(\boldsymbol{x}) \end{pmatrix} \sigma_3 \,, \qquad W_\ell^{-1}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{y}_\ell(\boldsymbol{x}) & \boldsymbol{z}_\ell(\boldsymbol{x}) \end{pmatrix} \,, \qquad (25)$$

$$W_0(\boldsymbol{x}) = \sigma_1 \begin{pmatrix} \boldsymbol{y}_0^\dagger(\boldsymbol{x}) \\ \boldsymbol{y}_{-1}^\dagger(\boldsymbol{x}) \end{pmatrix} \sigma_3 \,, \qquad W_0^{-1}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{y}_0(\boldsymbol{x}) & \boldsymbol{y}_{-1}(\boldsymbol{x}) \end{pmatrix} \,, \qquad (26)$$

with $\Lambda = \ell, 0$.

The completeness condition,

$$\sum_\ell \left[\boldsymbol{y}_\ell(\boldsymbol{x})\boldsymbol{y}_\ell^\dagger(\boldsymbol{x}') - \boldsymbol{z}_\ell(\boldsymbol{x})\boldsymbol{z}_\ell^\dagger(\boldsymbol{x}') \right]$$
$$+ \boldsymbol{y}_0(\boldsymbol{x})\boldsymbol{y}_{-1}^\dagger(\boldsymbol{x}') + \boldsymbol{y}_{-1}(\boldsymbol{x})\boldsymbol{y}_0^\dagger(\boldsymbol{x}') = \sigma_3\delta(\boldsymbol{x} - \boldsymbol{x}') \,, \qquad (27)$$

is simply expressed as

$$\sum_\ell W_\Lambda^{-1}(\boldsymbol{x}) \, W_\Lambda(\boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \,, \qquad (28)$$

and the field operators in the doublet form are expanded as

$$\varphi^\alpha(x) = \sum_\Lambda W_\Lambda^{-1,\alpha\beta}(\boldsymbol{x})b_\Lambda^\beta(t) \,, \qquad (29)$$

$$\bar{\varphi}^\beta(x) = \sum_\Lambda \bar{b}_\Lambda^\alpha(t)W_\ell^{\alpha\beta}(\boldsymbol{x}) \,, \qquad (30)$$

where

$$b_\ell^\alpha = \begin{pmatrix} b_\ell \\ b_\ell^\dagger \end{pmatrix}^\alpha \,, \qquad \bar{b}_\ell^\alpha = \begin{pmatrix} b_\ell^\dagger & -b_\ell \end{pmatrix}^\alpha \,, \qquad (31)$$

$$b_0^\alpha = \begin{pmatrix} -iq \\ p \end{pmatrix}^\alpha \,, \qquad \bar{b}_\ell^\alpha = \begin{pmatrix} p & iq \end{pmatrix}^\alpha \,. \qquad (32)$$

The operators b_ℓ, p, and q satisfy the canonical commutation relations $[b_\ell, b_{\ell'}^\dagger] = \delta_{\ell\ell'}$ and $[q, p] = i$, respectively, and the unperturbed Hamiltonian becomes

$$H_0^\varphi = H_0 = \frac{1}{2}\int d^3x \; \bar{\varphi}^\alpha(x) \, T^{\alpha\beta}(\boldsymbol{x}) \, \varphi^\beta(x) \qquad (33)$$

$$= \sum_\ell \omega_\ell b_\ell^\dagger b_\ell + \frac{p^2}{2} \,, \qquad (34)$$

which is diagonalized except for the zero mode sector, that is, that of the quantum mechanical operators p and q. The choice of the wave function for the sector is not settled yet. Although the diagrammatic calculation is possible once a particular wave function for the sector is chosen, we suppress the zero mode in what follows in order to avoid the ambiguity.

B. Field expansion for time-dependent order parameter

The time-dependent order parameter $\zeta(x)$ implies the time-dependent $T(x)$. In order to deal with this situation, we suppose a time-dependent orthonormal complete set $\{W_\ell(x)\}$ which is defined by Eq. (25) with some time-dependent functions $\boldsymbol{y}_\ell(x)$ and $\boldsymbol{z}_\ell(x)$ and which has the properties of Eqs. (24) and (28) at equal time. Then the field operators are expanded as

$$\varphi^\alpha(x) = \sum_\ell W_\ell^{-1,\alpha\beta}(x) b_\ell^\beta(t) \,. \tag{35}$$

We note that because of the time-dependence of $W_\ell(x)$ the time-evolution operator for for b_ℓ is not H_0^φ but H_0^b:

$$H_0^b(t) = H_0^\varphi(t)$$
$$- \frac{i}{2} \int d^3x d^3x' \, \bar{\varphi}^\alpha(x) \left[\sum_\ell \dot{W}_\ell^{-1}(x) W_\ell(x') \right]^{\alpha\beta} \varphi^\beta(x') \tag{36}$$

Here and hereafter, we take a common time variable for $x = (\boldsymbol{x}, t)$ and $x' = (\boldsymbol{x}', t)$. As we are going to treat the time-dependent order parameter in the quasi-particle picture represented by b_ℓ, the necessary condition is that H_0^b to be diagonal:

$$H_0^b(t) = \sum_\ell \lambda_\ell(t) b_\ell^\dagger(t) b_\ell(t) \tag{37}$$

$$= \frac{1}{2} \sum_\ell \bar{b}_\ell^\alpha(t) \left[\lambda_\ell(t) \sigma_3 \right]^{\alpha\beta} b_\ell^\beta(t) \,, \tag{38}$$

where $\lambda_\ell(t)$ is an arbitrary real function. Therefore, $W_\ell(x)$ must satisfy

$$T W_\ell^{-1} = i \frac{\partial}{\partial t} W_\ell^{-1} + \lambda_\ell W_\ell^{-1} \sigma_3 \,. \tag{39}$$

We eliminate λ_ℓ from the equations by the replacement $W_\ell^{-1}(x) \to W_\ell^{-1}(x) \, e^{i \int^t ds \, \lambda_\ell(s) \sigma_3}$, and obtain

$$T(x) W_\ell^{-1}(x) = i \frac{\partial}{\partial t} W_\ell^{-1}(x) \,, \tag{40}$$

or equivalently the TDBdG equations

$$T(x) \boldsymbol{y}_\ell(x) = i \frac{\partial}{\partial t} \boldsymbol{y}_\ell(x) \,. \tag{41}$$

If the orthonormal complete set $\{W_\ell(x)\}$ is chosen as the initial condition of Eq. (40), it keeps the orthonormality and the completeness for all the time because of the following equations:

$$i \frac{d}{dt} \int d^3x \, W_\ell(x) W_{\ell'}^{-1}(x) = 0 \,, \tag{42}$$

$$i \frac{\partial}{\partial t} \sum_\ell W_\ell^{-1}(x) W_\ell(x') = 0 \,. \tag{43}$$

Thus the procedure of constructing the time-dependent complete set is obtained: Solve the eigenvalue problem $T(x) W_\ell(x) = \omega_\ell W_\ell(x)$ at an initial time t_0, and calculate the time evolution according to $i \frac{\partial}{\partial t} W_\ell(x) = T(x) W_\ell(x)$. Next, expand the field operator $\varphi(x)$ by the complete set $\{W_\ell(x)\}$ as Eq. (35). This expansion is obviously reduced to the ordinary one in the limit of time-independent order parameter and has already been proposed by Matsumoto and Sakamoto [25]. What we have shown in the above paragraph is that the choice of $W_\ell(x)$ is justified from the viewpoint of quantum field theory: the quasi-particle operator b_ℓ, constructing the Fock space on the the Bose–Einstein condensed vacuum, diagonalizes the time evolution operator even for the time-dependent order parameter.

III. NONEQUILIBRIUM TFD FORMULATION

In this section, we double every degree of freedom to treat a nonequilibrium system in TFD and introduce the thermal Bogoliubov transformation

$$\begin{pmatrix} b_\ell \\ \tilde{b}_\ell^\dagger \end{pmatrix} = B_\ell^{-1} \begin{pmatrix} \xi_\ell \\ \tilde{\xi}_\ell^\dagger \end{pmatrix} \,, \qquad \left(b_\ell^\dagger \; -\tilde{b}_\ell \right) = \left(\xi_\ell^\dagger \; -\tilde{\xi}_\ell \right) B_\ell \,, \tag{44}$$

with

$$B_\ell^{\mu\nu}(t) = \begin{pmatrix} 1 + n_\ell(t) & -n_\ell(t) \\ -1 & 1 \end{pmatrix}^{\mu\nu} \,, \tag{45}$$

$$B_\ell^{-1,\mu\nu}(t) = \begin{pmatrix} 1 & n_\ell(t) \\ 1 & 1 + n_\ell(t) \end{pmatrix}^{\mu\nu} \,. \tag{46}$$

It is important to take the above particular form of the thermal Bogoliubov matrix, as one calls the $\alpha = 1$ representation [16], which enables us to make use of the Feynman diagram method in nonequilibrium systems [26]. In TFD, the thermal average is represented by the pure state expectation of the thermal vacuum, denoted by $|0\rangle$, and the operators who annihilate $|0\rangle$, are not the b-operators but the ξ-ones:

$$\xi_\ell |0\rangle = \tilde{\xi}_\ell |0\rangle = 0 \,, \qquad \langle 0| \xi_\ell^\dagger = \langle 0| \tilde{\xi}_\ell^\dagger = 0 \,. \tag{47}$$

The number distribution $n_\ell(t)$ is given by

$$n_\ell(t) = \langle 0| b_\ell^\dagger(t) b_\ell(t) |0\rangle \,, \tag{48}$$

and its time dependence is determined later. The combination of the two transformations, ξ into b and b into φ, involves the 4×4-matrix transformations,

$$b_\ell^{\mu\alpha} = \mathcal{B}_\ell^{-1,\mu\alpha\nu\beta} \xi_\ell^{\nu\beta} \,, \qquad \bar{b}_\ell^{\nu\beta} = \xi_\ell^{\mu\alpha} \mathcal{B}_\ell^{\mu\alpha\nu\beta} \,, \tag{49}$$

$$\varphi^{\mu\alpha} = \sum_\ell \mathcal{W}_\ell^{-1,\mu\alpha\nu\beta} b_\ell^{\nu\beta} \,, \quad \bar{\varphi}^{\nu\beta} = \sum_\ell \bar{b}_\ell^{\mu\alpha} \mathcal{W}_\ell^{\mu\alpha\nu\beta} \,, \tag{50}$$

with the 4×4 thermal Bogoliubov and BdG matrices

$$\mathcal{B}_\ell^{-1,\mu\alpha\nu\beta} = \delta_{\alpha 1}\delta_{\beta 1}B_\ell^{-1,\mu\nu} + \delta_{\alpha 2}\delta_{\beta 2}\left(\sigma_1 B_\ell^{-1}\sigma_1\right)^{\mu\nu}, \quad (51)$$

$$\mathcal{B}_\ell^{\mu\alpha\nu\beta} = \delta_{\alpha 1}\delta_{\beta 1}B_\ell^{\mu\nu} + \delta_{\alpha 2}\delta_{\beta 2}\left(\sigma_1 B_\ell \sigma_1\right)^{\mu\nu}, \quad (52)$$

$$\mathcal{W}_\ell^{-1,\mu\alpha\nu\beta} = \delta_{\mu\nu}W_\ell^{-1,\alpha\beta}, \quad (53)$$

$$\mathcal{W}_\ell^{\mu\alpha\nu\beta} = \delta_{\mu\nu}W_\ell^{\alpha\beta}, \quad (54)$$

where the quartet notations for b_ℓ are introduced by

$$b_\ell^{\mu\alpha} = \begin{pmatrix} b_\ell^\mu \\ [\sigma_1 \tilde{b}_\ell]^\mu \end{pmatrix}^\alpha = \begin{pmatrix} b_\ell \\ \tilde{b}_\ell^\dagger \\ b_\ell^\dagger \\ \tilde{b}_\ell \end{pmatrix}^{\mu\alpha}, \quad (55)$$

$$\bar{b}_\ell^{\nu\beta} = \left(\bar{b}_\ell^\nu \;\; [\tilde{\bar{b}}_\ell \sigma_1]^\nu\right)^\beta = \left(b_\ell^\dagger \;\; -\tilde{b}_\ell \;\; -b_\ell \;\; \tilde{b}_\ell^\dagger\right)^{\nu\beta}, \quad (56)$$

and in similar fashions for ξ_ℓ and φ. The Hamiltonian of TFD, which should generate the time translations of both the non-tilde and tilde operators, is not the ordinary Hamiltonian H but the hat Hamiltonian $\hat{H} = H - \tilde{H}$. The time-independence of the thermal vacua requires the minus sign in front of $\tilde{H}$. Furthermore, the unperturbed Hamiltonian in nonequilibrium TFD is $\hat{H}_Q = \hat{H}_0 - \hat{Q}$ for nonequilibrium system with $\hat{H}_0 = H_0 - \tilde{H}_0$ and the thermal counter term $\hat{Q}(t)$, caused by the time-dependence of $n_\ell(t)$:

$$\hat{Q}(t) = -\frac{i}{2}\sum_\ell \dot{n}_\ell(t)\,\bar{\xi}_\ell^{\mu\alpha}(t) \begin{pmatrix} \begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & 1 & 0 \end{array} \end{pmatrix}^{\mu\alpha\nu\beta} \xi_\ell^{\nu\beta}(t). \quad (57)$$

The unperturbed and full propagators for φ and ξ are

defined by

$$\Delta^{\mu\alpha\nu\beta}(x_1, x_2) = -i\langle 0|T[\varphi^{\mu\alpha}(x_1)\bar{\varphi}^{\nu\beta}(x_2)]|0\rangle, \quad (58)$$

$$G^{\mu\alpha\nu\beta}(x_1, x_2) = -i\langle 0|T[\varphi_\mathrm{H}^{\mu\alpha}(x_1)\bar{\varphi}_\mathrm{H}^{\nu\beta}(x_2)]|0\rangle, \quad (59)$$

$$d_{\ell\ell'}^{\mu\alpha\nu\beta}(t_1, t_2) = -i\langle 0|T[\xi_\ell^{\mu\alpha}(t_1)\bar{\xi}_{\ell'}^{\nu\beta}(t_2)]|0\rangle, \quad (60)$$

$$g_{\ell\ell'}^{\mu\alpha\nu\beta}(t_1, t_2) = -i\langle 0|T[\xi_{\mathrm{H}\ell}^{\mu\alpha}(t_1)\bar{\xi}_{\mathrm{H}\ell'}^{\nu\beta}(t_2)]|0\rangle, \quad (61)$$

respectively, where the subscript H denotes a quantity in the Heisenberg picture. The self-energies Σ and S are introduced by the Dyson equations $G = \Delta + \Delta\Sigma G$ and $g = d + dSg$, respectively.

Our critical step is to adopt the following three renormalization conditions simultaneously to determine the whole time evolution of the system, explicitly to determine the unknown functions $n_\ell(t)$, $\delta T(x)$ and $\zeta(x)$ (or $\delta C(x)$, see Eq. (16)):

$$\begin{array}{lll} \text{(i)} & g_{\ell\ell}^{1121}(t,t) = 0 & \\ \text{(ii)} & \mathrm{Re}\, S_{\ell\ell,\text{on-shell}}^{1111} = 0, & \\ \text{(iii)} & \langle 0|\varphi_\mathrm{H}(x)|0\rangle = 0. & \end{array}$$

The condition (i) is what we have proposed for a nonequilibrium system with the static condensate [19] as a natural extension of the one for non-condensed system proposed by Chu and Umezawa [17, 18, 27]. It provides the transport equation which determines the temporal evolution of the unperturbed number distribution $n_\ell(t)$. The possible diagrams in the leading order are indicated in Fig. 1. Because the contributions from Fig. 1 (a) and (c) vanish, the leading ones come from Fig. 1 (b) and (d), and the latter one is proportional to $\dot{n}_\ell(t)$. According to the detailed calculations given in Ref. [19], we obtain

$$\dot{n}_\ell(t) = 4g^2 \mathrm{Re}\int_{-\infty}^t dt_1 \sum_{\ell_1\ell_2}\Big[\;\; \{n_{\ell_1}n_{\ell_2}(1+n_\ell) - (1+n_{\ell_1})(1+n_{\ell_2})n_\ell\}_{t_1}(\boldsymbol{y}_\ell, \boldsymbol{\chi}_{yy})_t(\boldsymbol{\chi}_{yy}, \boldsymbol{y}_\ell)_{t_1}$$
$$+ \{n_{\ell_1}(1+n_{\ell_2})(1+n_\ell) - (1+n_{\ell_1})n_{\ell_2}n_\ell\}_{t_1}(\boldsymbol{y}_\ell, \boldsymbol{\chi}_{yz})_t(\boldsymbol{\chi}_{yz}, \boldsymbol{y}_\ell)_{t_1}$$
$$+ \{(1+n_{\ell_1})n_{\ell_2}(1+n_\ell) - n_{\ell_1}(1+n_{\ell_2})n_\ell\}_{t_1}(\boldsymbol{y}_\ell, \boldsymbol{\chi}_{zy})_t(\boldsymbol{\chi}_{zy}, \boldsymbol{y}_\ell)_{t_1}$$
$$+ \{(1+n_{\ell_1})(1+n_{\ell_2})(1+n_\ell) - n_{\ell_1}n_{\ell_2}n_\ell\}_{t_1}(\boldsymbol{y}_\ell, \boldsymbol{\chi}_{zz})_t(\boldsymbol{\chi}_{zz}, \boldsymbol{y}_\ell)_{t_1}\Big], \quad (62)$$

where

$$\chi_{yy}^\alpha = \zeta^{\bar{\alpha}}y_{\ell_1}^\alpha y_{\ell_2}^\alpha + \zeta^\alpha y_{\ell_1}^{\bar{\alpha}} y_{\ell_2}^\alpha + \zeta^\alpha y_{\ell_1}^\alpha y_{\ell_2}^{\bar{\alpha}}, \quad (63)$$

$$\chi_{yz}^\alpha = \zeta^{\bar{\alpha}}y_{\ell_1}^\alpha z_{\ell_2}^\alpha + \zeta^\alpha y_{\ell_1}^{\bar{\alpha}} z_{\ell_2}^\alpha + \zeta^\alpha y_{\ell_1}^\alpha z_{\ell_2}^{\bar{\alpha}}, \quad (64)$$

$$\chi_{zy}^\alpha = \zeta^{\bar{\alpha}}z_{\ell_1}^\alpha y_{\ell_2}^\alpha + \zeta^\alpha z_{\ell_1}^{\bar{\alpha}} y_{\ell_2}^\alpha + \zeta^\alpha z_{\ell_1}^\alpha y_{\ell_2}^{\bar{\alpha}}, \quad (65)$$

$$\chi_{zz}^\alpha = \zeta^{\bar{\alpha}}z_{\ell_1}^\alpha z_{\ell_2}^\alpha + \zeta^\alpha z_{\ell_1}^{\bar{\alpha}} z_{\ell_2}^\alpha + \zeta^\alpha z_{\ell_1}^\alpha z_{\ell_2}^{\bar{\alpha}}, \quad (66)$$

with $\zeta^\alpha(x) = \begin{pmatrix} \zeta(x) \\ \zeta^*(x) \end{pmatrix}^\alpha$, and $\bar{\alpha}$ denotes $\bar{\alpha} = 2, 1$ for

$\alpha = 1, 2$, respectively. The subscripts t and t_1 are the time arguments, and the parenthesis denotes the inner product:

$$(\boldsymbol{y}, \boldsymbol{\chi})_t = \int d^3x\, y^{*,\alpha}(x)\chi^\alpha(x). \quad (67)$$

The term in the fourth line of Eq. (62) is what we call the triple production term. As is discussed in Ref. [19], this term plays a crucial role if there is the Landau instability, but is vanishing due to the energy conservation otherwise.
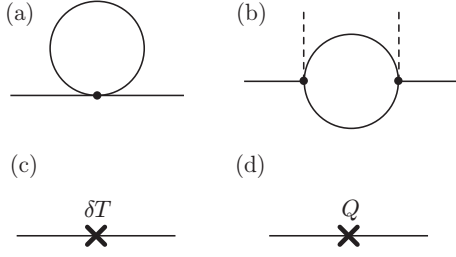
FIG. 1: One-loop propagators and counter terms. The solid and dashed lines denote the unperturbed propagator and the order parameter ζ, respectively.

While the transport equations with a phase-space distribution, derived in the other methods previously, lack this term, the prediction of our transport equation is different from those based on the other transport equations, when there is the Landau instability.

The condition (ii) is the energy renormalization which determines δT. Since the leading contribution to the self-energy is the tag diagram in Fig. 1(a) and has the form of $S_{\ell\ell'}(t_1, t_2) = S'_{\ell\ell'}(t_1)\delta(t_1 - t_2)$, the on-shell self-energy is naturally defined as $S'_{\ell\ell}(t_1)$ even in the nonequilibrium situation. The concrete forms for the self-energy of Fig. 1(a) and (c) are

$$S_{\ell\ell}^{(a),1111}(t_1, t_2) = \delta(t_1 - t_2)\big(\boldsymbol{y}_\ell, \sigma_3 T^{(a)}\boldsymbol{y}_\ell\big)_{t_1}, \qquad (68)$$

$$S_{\ell\ell}^{(c),1111}(t_1, t_2) = \delta(t_1 - t_2)\big(\boldsymbol{y}_\ell, \sigma_3 \delta T\boldsymbol{y}_\ell\big)_{t_1}, \qquad (69)$$

with

$$T^{(a),\alpha\beta}(x) = g\begin{pmatrix} 2\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle & \langle 0|\varphi(x)\varphi(x)|0\rangle \\ -\langle 0|\varphi(x)\varphi(x)|0\rangle^* & -2\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle \end{pmatrix}^{\alpha\beta}. \qquad (70)$$

Then, we find the condition (ii) is satisfied by

$$\delta T(x) = T^{(a)}(x), \qquad (71)$$

which is equivalent to the result of the Hartree-Fock-Bogoliubov approximation. The matrix elements of $T^{(a)}$

can be rewritten explicitly in terms of $n_\ell(t)$ and $\boldsymbol{y}_\ell(x)$ as

$$\langle 0|\varphi^\dagger\varphi|0\rangle = \sum_\ell \big[n_\ell \big(|y_\ell^1|^2 + |y_\ell^2|^2\big) + |y_\ell^2|^2\big], \qquad (72)$$

$$\langle 0|\varphi\varphi|0\rangle = \sum_\ell (2n_\ell + 1)y_\ell^1 y_\ell^{2,*}. \qquad (73)$$

One can show from Eq. (41) that the time-dependence of the total number of non-condensed atoms $N_{\text{ex}}(t) = \int d^3x\langle 0|\varphi^\dagger(x)\varphi(x)|0\rangle$ is

$$\frac{d}{dt}N_{\text{ex}}(t) = \sum_\ell \big[(2n_\ell(t) + 1)\text{Im}\big(\boldsymbol{y}_\ell, T_0\boldsymbol{y}_\ell\big) + \dot{n}_\ell(t)\big(\boldsymbol{y}_\ell, \boldsymbol{y}_\ell\big)\big], \qquad (74)$$

where $\delta T(x)$ drops because of $\text{Im}\big(\boldsymbol{y}_\ell, \delta T\boldsymbol{y}_\ell\big) = 0$. Note that the second term in Eq. (74) is of two-loop order,
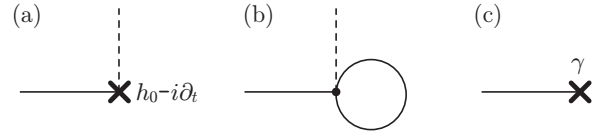

FIG. 2: Tadpole diagrams.

although we collect only the one-loop diagrams. This comes from the fact that $\dot{n}_\ell$ is of one-loop order according to the quantum transport equation (62), while n_ℓ itself is of no loop one.

The last condition (iii) is the self-consistent criterion for dividing the original field operator ψ into ζ and φ. The corresponding diagrams at tree and one-loop levels are shown in Fig. 2 (a) and (b), respectively. Although the energy renormalization is performed at one-loop level, the two-loop order correction indicated in Fig. 2 (c) is also considered here for the conservations of the total number of atoms. The quantity $\gamma(x)$ which is a two-loop order part of the counter term $\delta C(x)$ is determined later to cancel the two-loop contribution which appears in Eq. (74). The condition with the diagrams in Fig. 2 is written as

$$\int d^4x_1 \big(\Delta^{\mu\alpha 11}(x, x_1) + \Delta^{\mu\alpha 21}(x, x_1)\big)$$

$$\times \Big[\big\{h_0(x_1) - i\partial_{t_1} + 2g\langle 0|\varphi^\dagger(x_1)\varphi(x_1)|0\rangle\big\}\zeta(x_1) + g\langle 0|\varphi(x_1)\varphi(x_1)|0\rangle\zeta^*(x_1) - i\gamma(x_1)\Big]$$

$$- \int d^4x_1 \big(\Delta^{\mu\alpha 12}(x, x_1) + \Delta^{\mu\alpha 22}(x, x_1)\big)$$

$$\times \Big[\big\{h_0(x_1) + i\partial_{t_1} + 2g\langle 0|\varphi^\dagger(x_1)\varphi(x_1)|0\rangle\big\}\zeta^*(x_1) + g\langle 0|\varphi(x_1)\varphi(x_1)|0\rangle^*\zeta(x_1) + i\gamma(x_1)\Big] = 0, \qquad (75)$$

which implies

$$i\frac{\partial}{\partial t}\zeta = \big(h_0 + 2g\langle 0|\varphi^\dagger\varphi|0\rangle\big)\zeta + g\langle 0|\varphi\varphi|0\rangle\zeta^* - i\gamma, \quad (76)$$

where h_0 is defined in Eq. (10). Thus, the modified

TDGP equation has been derived. To find the function $\gamma(x)$, we employ the Φ derivative approximation [28, 29], which can derive the conserving TDGP equation. In this approximation, the self-energy is redefined by the derivative of the functional $\Phi = \Phi[G]$ as

$$\Sigma(x_1, x_2) = \frac{\delta\Phi}{\delta G(x_1, x_2)} \,. \qquad (77)$$

Because the two-loop order modification caused by $\dot{n}_\ell$, or the thermal counter term $\hat{Q}$ in other words, is required here, we only calculate the part of Φ which is related to $\hat{Q}$ in the leading order. The contribution of the thermal counter term to the self-energy is

$$\Sigma_Q^{\mu\alpha\nu\beta}(x_1, x_2) = \sum_{\ell_1, \ell_2} \Big[\mathcal{W}_{\ell_1}^{-1}(x_1)\mathcal{B}_{\ell_1}^{-1}(t_1)$$
$$\times S_{Q,\ell_1\ell_2}(t_1, t_2)\mathcal{B}_{\ell_2}(t_2)\mathcal{W}_{\ell_2}(x_2)\Big]^{\mu\alpha\nu\beta} \quad (78)$$

with

$$S_{Q,\ell_1\ell_2}^{\mu\alpha\nu\beta}(t_1, t_2) = -i\dot{n}_{\ell_1}(t_1)\delta(t_1-t_2)\delta_{\ell_1\ell_2}\begin{pmatrix} \begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & 1 & 0 \end{array} \end{pmatrix}^{\mu\alpha\nu\beta},$$
$$(79)$$

and that to the functional Φ is

$$\Phi_Q = \int d^4x_1 d^4x_2 \, \Sigma_Q^{\mu\alpha\nu\beta}(x_1, x_2)\Delta^{\nu\beta\mu\alpha}(x_2, x_1)\varepsilon^\mu\varepsilon^\alpha$$
$$(80)$$
$$= -\sum_\ell \int dt \, \dot{n}_\ell(t)\big(\boldsymbol{y}_\ell, \boldsymbol{y}_\ell\big)_t \,, \qquad (81)$$

with the sign factor, $\varepsilon^1 = 1$ and $\varepsilon^2 = -1$. According to the Φ derivative approximation, $\gamma(x)$ which is the two-loop correction involved in Σ_Q is found to be

$$\gamma(x) = \frac{\delta\Phi_Q}{\delta\zeta^*(x)} = -\sum_\ell \left[\frac{\delta\dot{n}_\ell(t)}{\delta\zeta^*(x)} \big(\boldsymbol{y}_\ell, \boldsymbol{y}_\ell\big)_t \right] \,. \qquad (82)$$

By substituting Eq. (62) into this, we obtain

$$\gamma(x) = g^2\mathrm{Re}\int_{-\infty}^t dt_1 \sum_{\ell_1\ell_2\ell_3} \big(\boldsymbol{y}_{\ell_3}, \boldsymbol{y}_{\ell_3}\big)_t \bigg[$$
$$\{n_1 n_2(1+n_3) - (1+n_1)(1+n_2)n_3\}_{t_1} X_{yyz}^1(x)\big(\boldsymbol{\chi}_{yy}, \boldsymbol{y}_3\big)_{t_1}$$
$$+ \{n_1(1+n_2)(1+n_3) - (1+n_1)n_2 n_3\}_{t_1} X_{yzz}^1(x)\big(\boldsymbol{\chi}_{yz}, \boldsymbol{y}_3\big)_{t_1}$$
$$+ \{(1+n_1)n_2(1+n_3) - n_1(1+n_2)n_3\}_{t_1} X_{zyz}^1(x)\big(\boldsymbol{\chi}_{zy}, \boldsymbol{y}_3\big)_{t_1}$$
$$+ \{(1+n_1)(1+n_2)(1+n_3) - n_1 n_2 n_3\}_{t_1} X_{zzz}^1(x)\big(\boldsymbol{\chi}_{zz}, \boldsymbol{y}_3\big)_{t_1} \bigg], \qquad (83)$$

where

$$X_{yyz}^\alpha = y_1^\alpha y_2^\alpha z_3^{\bar{\alpha}} + y_1^\alpha y_2^{\bar{\alpha}} z_3^\alpha + y_1^{\bar{\alpha}} y_2^\alpha z_3^\alpha \,, \qquad (84)$$

and in similar fashions for X_{yzz}^α, X_{zyz}^α, and X_{zzz}^α. Then, the time derivative of the total number of condensed atoms $N_0(t) = \int d^3x \, |\zeta(x)|^2$ becomes

$$\frac{d}{dt}N_0 = -\sum_\ell \big[(2n_\ell + 1)\mathrm{Im}\big(\boldsymbol{y}_\ell, T_0\boldsymbol{y}_\ell\big) + \dot{n}_\ell\big(\boldsymbol{y}_\ell, \boldsymbol{y}_\ell\big) \big] \,, \qquad (85)$$

which cancels Eq. (74) and implies the conservation of the total atom number.

Thus, we obtain the coupled equations which describe the temporal evolution of the condensed system, those

are the TDGP equation (76), the TDBdG equations (41), and the quantum transport equation (62).

IV. NUMERICAL RESULT

In this section, we calculate the coupled equations numerically to illustrate the nonequilibrium dynamics, especially the condensate decays with either the Landau instability or the dynamical one, and confirm the qualitative difference between both the instabilities. For this propose, we consider a very simple system with the one-

dimensional Bose–Hubbard model with the Hamiltonian

$$H = \sum_i \left[-J\psi_i^\dagger \{\psi_{i+1} + \psi_{i-1}\} - \mu\psi_i^\dagger\psi_i + \frac{U}{2}\psi_i^\dagger\psi_i^\dagger\psi_i\psi_i \right].$$
(86)

Here J, U, and i represent the inter-site hopping, the on-site couping, and the site index, and we put the number of sites $I_s = 21$, the total number of atoms $N = 210$, and $U/J = 0.05$. The condensate is introduced as $\psi_i = \zeta_i + \varphi_i$ with the criterion $\langle 0|\varphi_i|0\rangle = 0$. It is straightforward to apply the method developed in the previous section to this model and to derive the coupled equations for this system in the leading order.

To illustrate the condensate decay with the Landau instability or the dynamical one, we consider the following situation. First, the equilibrium state with no condensate flow at a temperature T_0 is prepared. Then, the condensate is forced to flow instantaneously with the quasimomentum k : $\zeta_i \to \zeta_i e^{ikx_i}$, and the system turns into nonequilibrium. This nonequilibrium state is chosen as the initial state of the calculation, and then the coupled equations are calculated numerically.

Solving the BdG eigenvalue equations at zero temperature analytically, we obtain the stability diagram as Fig. 3 and find that the system is stable for $kL/2\pi = 0$ and 1, Landau unstable for $kL/2\pi = 2$ to 5, and dynamically unstable for $kL/2\pi = 6$ to 9. Although the result is for the zero temperature, it is also expected to be valid for the nonequilibrium case with a sufficiently small initial depletion.
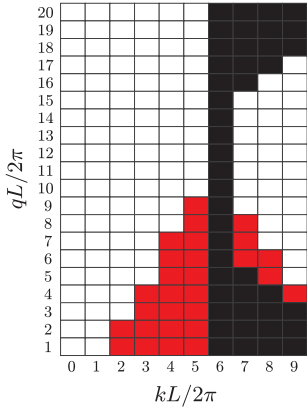

FIG. 3: Stability diagram for $U/J = 0.05$ obtained analytically from BdG eigenvalue equation at tree-level. The symbols k and q stand for the quasi-momenta of the condensate and the excitation, respectively, and L is the length of the system. The white, red, and black colored cells denote stable, Landau unstable, and dynamically unstable regions, respectively.

The coupled equations are calculated numerically with several initial conditions. The depletion of the condensate is indicated in Fig. 4, and the stable and unstable behaviors are clearly discriminated. For the stable condition $kL/2\pi = 1$, the nonequilibrium depletion oscillates around the equilibrium one and never grows. The depletion grows initially with oscillation for the Landau unsta-

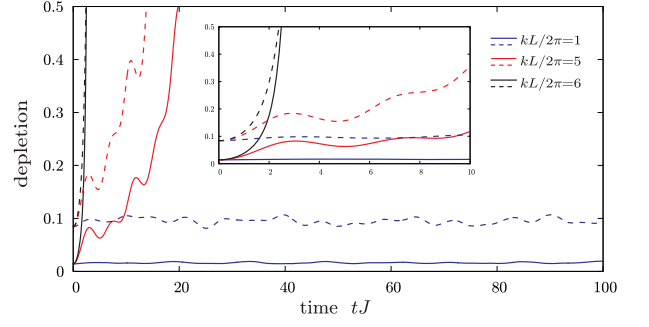FIG. 4: Temporal evolution of the depletion $\frac{1}{N}\sum_i \langle 0|\varphi_i^\dagger\varphi_i|0\rangle$. Solid (dashed) line indicates the result with smaller (larger) initial depletion, corresponding to a lower (higher) T_0. Blue, red, and black lines correspond to the stable, the Landau unstable, and the dynamically unstable conditions, respectively.
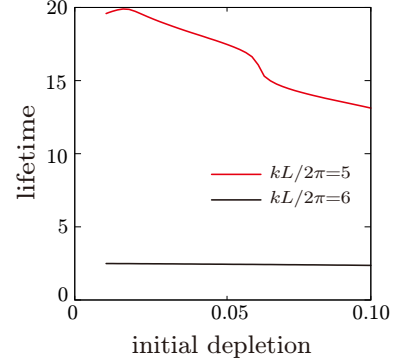

FIG. 5: Lifetime of the condensate vs initial depletion. Red and black lines correspond to Landau and dynamically unstable conditions, respectively. The lifetime is simply defined by the time when the depletion reaches to 0.5. The nonmonotonicity of the lifetime in case of the Landau instability is due to the oscillating growth of the depletion.

ble condition $kL/2\pi = 5$, but grows much more rapidly and exponentially from the beginning for the dynamically unstable one $kL/2\pi = 6$. The decay speed with the Landau instability tends to increase if the initial depletion becomes large as is shown in Fig. 5, reflecting the fact that the Landau instability is caused by the collision be-
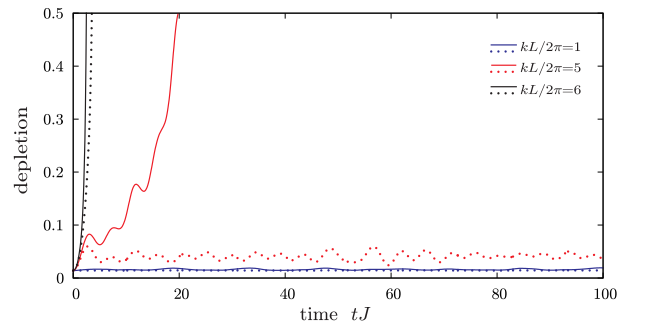

FIG. 6: The effect of the triple production term to the depletion growth. Solid line is the same as in Fig. 4, while doted line indicates the result, obtained when the triple production term is omitted.

tween condensate and non-condensate particles and that the collision becomes more frequent for larger depletion. On the other hand, the value of the initial depletion is not relevant for the dynamical instability, since the non-condensate particle plays no essential role then.

The effect of the triple production term which is a distinguishing feature of our quantum transport equation is shown in Fig. 6. No qualitative difference between the cases with the triple production term and without it is notable in both the stable and dynamically unstable conditions. That is because that the triple production is suppressed due to the energy conservation in the former, and that the rapidly growth is governed by the TDGP and TDBdG equations but not by the transport equation in the latter. In the Landau instability condition, on the other hand, the difference is remarkable. Basically, the growth behavior disappears if the triple production term is omitted. Thus we conclude the essential contribution of the triple production term in describing the Landau instability.

Although we present only the results of the three typical values of k, representing the stable, Landau unstable, and dynamically unstable situations, respectively, the qualitatively similar results are obtained for any allowed value of k.

V. SUMMARY

In this paper, the self-consistent equations which describe the nonequilibrium dynamics have been derived by applying the nonequilibrium TFD to the condensed atom system. The system with a time-dependent order parameter is considered as the further extension of our previous study where the stationary order parameter has been assumed. We only have been able to describe the first stage of the condensate decay with the Landau instability heretofore. Now we can predict the next stage of the decay dynamics, and can describe not only the Landau instability but also the dynamical one.

To treat the time-dependence of the order parameter within the time-independent quasiparticle picture, the field operator is expanded with the complete set evaluating by the TDBdG equations. This method, proposed first by Matsumoto and Sakamoto without a detailed vindication, is obtained here to construct the particle operator time-independent. The renormalization conditions, the the Chu-Umezawa's diagonalization condition (i), the

energy renormalization (ii), and the criterion for diving the non-condensate and the condensate (iii), are applied to determine the coupled equations which are the quantum transport equation, the TDBdG equations, and the TDGP equation.

The point is that the coupled equations we obtained are not the naive assemblages of presumable equations, but are the self-consistent ones derived by the appropriate renormalization conditions. To confirm that our coupled equations can describe both Landau and dynamical instability, we consider an one-dimensional Bose–Hubbard model, and calculate numerically the depletion whose growth implies the decay of the condensate. Since the initial depletion is sufficiently small, the unstable condition is well characterized by the eigenvalues of the BdG equations at tree-level. We found that the depletion growths with both the Landau and dynamically unstable condition, while dose not with stable condition. Predictably in the Landau unstable condition, the growth is slower and has stronger initial depletion dependence than that in the dynamically unstable condition.

As we reported previously, our transport equation contains the triple production term which is absent in the one of the other methods. The difference originates in the choice of the quasiparticle picture which is essential for the quantum field theory. We construct the quasiparticle faithfully to the quantum field theory, while the others are based on a phase-space distribution function and no explicit particle representation is given. This difference is inconspicuous if there is no Landau instability, but causes a great qualitative change if there is Landau instability. Basically, we find that the condensate dose not decay without the triple production term in the Landau unstable condition, although the decay has been observed experimentally. We emphasize that the choice of the quasiparticle picture is even more essential for a unstable case. A quantitative comparison to a experiment of the condensate decay will be the future task.

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